Finite temperature Casimir effect in the presence of nonlinear dielectrics

Fardin Kheirandish,[1,*] Ehsan Amooghorban,[1,†] and Morteza Soltani[1,‡]

[1] *Department of Physics, Faculty of Science,*
University of Isfahan, Isfahan, Iran.

Abstract

Starting from a Lagrangian, electromagnetic field in the presence of a nonlinear dielectric medium is quantized using path-integral techniques and correlation functions of different fields are calculated. The susceptibilities of the nonlinear medium are obtained and their relation to coupling functions are determined. Finally, the Casimir energy and force in the presence of a nonlinear medium at finite temperature is calculated.

PACS numbers: 12.20.Ds, 03.70.+k, 42.50.Nn

*fardin˙kh@phys.ui.ac.ir
†amooghorban@sci.ui.ac.ir
‡msoltani@phys.ui.ac.ir

I. INTRODUCTION

One of the most direct manifestations of the zero-point vacuum oscillations is the Casimir effect. This effect in its simplest form, is the attraction force between two neutral, infinitely large, parallel and ideal conductors in vacuum. The effect is completely quantum mechanical and is a result of electromagnetic field quantization in the presence of some boundary conditions. The presence or absence of boundary conditions cause a finite change of vacuum-energy which its variation respect to the distance between the conductors gives the Casimir force [1]. The general theory of forces between parallel dielectrics was worked out by Lifshitz et al [2]. Some years later, the theory of the Casimir effect was rederived by Schwinger [3] and later on this theory extended to systems consisting of some dielectric layers [4]-[5] and precision experiments on measuring the Casimir force performed [6]-[9].

Generally, calculations of Casimir forces are based on two different approaches [10]. In the first approach, the total energy of discrete quantum modes of the electromagnetic field is calculated and usually a regularization procedure is required, although there is still no consensus on the regularization procedures, which often lead to inconsistent results. On the other hand, this formalism despite of its simplicity, is restricted to systems where energy eigenvalues are known, but geometries in which energy eigenvalues are known exactly are few.

The second approach is based on a Greens function method where using the fluctuationdissipation theorem, the electromagnetic field energy density is linked directly to the photonic Greens function.

In recent years considerable attention has paid to the Casimir effect because of its wide applications in different areas of physics such as quantum field theory, gravitation and cosmology, atomic physics, condensed matter, nanotechnology, and mathematical physics [11]-[14]. On the other hand, during the last two decades, the emergence of periodically structured optical materialscommonly called photonic crystals- has lead to substantial progress in the science and technology of optics and photonics [15]. Photonic crystals [16]-[17] provide new opportunities for enhancing and controlling nonlinear optical processes. An important step in this direction is to provide a quantum theory of light that takes into account nonlinear processes and at the same time include the absorption in the nonlinear material. Several attempts have been made to provide a quantized theory of macroscopic nonlinear electro-

dynamics [18]-[23] and most of them are focused on strictly lossless materials. Recently, a consistent approach that includes absorption and dispersion has been formulated within the frame work of quantum brownian motion [24]. This motivated us to investigate the quantization of electromagnetic field in the presence of a nonlinear dielectric medium and also consider the nonlinear effects of the medium on Casimir effect which can have applications in both fundamental science and engineering.

In the 1990s, Golestanian and Kardar developed a path-integral approach to investigate the dynamic Casimir effect in a system which was consisted of two corrugated conducting plates surrounded by quantum vacuum [25]-[26]. Emig and his colleagues also used the path-integral formalism to obtain normal and lateral Casimir force between two sinusoidal corrugated perfect conductor surfaces [27]-[28]. Recently, this formalism is merged with a canonical approach to calculate the Casimir force between two perfectly conducting plates immersed in a magnetodielectric medium [29]. In the present work the electromagnetic field in the presence of a nonlinear magnetodielectric medium is quantized in the framework of path-integrals and as an example the mutual Casimir force between conducting parallel plates, immersed in a nonlinear dielectric medium, is calculated.

The layout of the present work is as follows: In Sec.II, a Lagrangian for electromagnetic field in the presence of a nonlinear medium is proposed and quantization is achieved via path-integrals. In Secs.III and IV, the linear and nonlinear green's functions of the electromagnetic field and medium are obtained and the nth order susceptibility of the nonlinear medium which satisfies the kramers kronig relations is determined. In Sec.V, Casimir force in the presence of a linear and nonlinear medium for zero and finite temperature is calculated. Finally, we discuss the main results and conclude in Sec.VI.

II. FIELD QUANTIZATION

Quantum electrodynamics in a linear magnetodielectric medium can be accomplished by modeling the medium with two independent reservoirs interacting with the electromagnetic field. Each reservoir contains a continuum of three dimensional harmonic oscillators describing the electric and magnetic properties of the medium [30]-[32]. In this section we follow the idea introduced in [24] to quantize the electromagnetic field in the presence of a nonlinear medium and for simplicity we restrict our attention to a nonmagnetic medium

but the generalization to a magnetodielectric medium is straightforward. For this purpose let us consider the following total Lagrangian density

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_{mat} + \mathcal{L}_{int} \tag{1}$$

where $\mathcal{L}_{EM}$ is electromagnetic field Lagrangian density

$$\mathcal{L}_{EM} = \frac{1}{2}\varepsilon_0 \mathbf{E}^2 - \frac{\mathbf{B}^2}{2\mu_0}. \tag{2}$$

The physical fields can be written in terms of the potentials as $\mathrm{E} = -\frac{\partial \mathrm{A}}{\partial t} - \nabla\phi$ and $\mathrm{B} = \nabla \times \mathrm{A}$, and for simplicity we work in the axial gauge where $\phi = 0$. The dielectric medium is modeled by a vector field $\mathbf{X}(\omega)$, which is suppose to describe its electrical properties

$$\mathcal{L}_{mat} = \int_0^\infty d\omega \frac{1}{2}\dot{\mathbf{X}}^2(\omega, x) - \frac{1}{2}\omega^2 \mathbf{X}^2(\omega, x). \tag{3}$$

Now we define i'th component of the polarization field of the dielectric medium as

$$\begin{aligned} P_i(x) \; &= \int_0^\infty d\omega \nu^{(1)}(\omega) X_i(\omega, x) + \int_0^\infty d\omega \int_0^\infty d\omega' \nu^{(2)}(\omega, \omega') X_i(\omega, x) X_i(\omega', x) \\ &+ \int_0^\infty d\omega \int_0^\infty d\omega' \int_0^\infty d\omega'' \nu^{(3)}(\omega, \omega', \omega'') X_i(\omega, x) X_i(\omega', x) X_i(\omega'', x) + ... \end{aligned} \tag{4}$$

In these expression, the index i can take on the values x, y and z and $\nu^{(1)}, \nu^{(2)}, \nu^{(3)}, \cdots,$ are the coupling functions between the medium and the electromagnetic field. As it can be seen from (4), the coupling-tensor $\nu^{(1)}$ describes the linear contribution of the interaction and the sequence $\nu^{(2)}, \nu^{(3)}, ...$ describe, respectively, the first-order, second-order and higher orders of non-linear interactions. The coupling-tensors $\nu^{(1)}, \nu^{(2)}, \nu^{(3)}, ...$ in (4) are the key parameters in this quantization scheme and we will see that the susceptibility functions of the medium can be expressed in terms of these coupling-tensors.

The interaction part of Lagrangian is defined by [24]

$$\mathcal{L}_{int} = \mathbf{A} \cdot \dot{\mathbf{P}}. \tag{5}$$

Now we quantize the theory using the path-integral techniques. For this purpose let us start with the following generating functional [33]

$$Z[J] = \int \mathcal{D}[\varphi] \exp \imath \int d^4x [\mathcal{L}(\varphi(x)) + J(x)\varphi(x)], \tag{6}$$

where φ is a scalar field and J acts as a source or an auxiliary field and different correlation functions can be found by taking repeated functional derivatives with respect to the field

$J(x)$. The above partition function is Gaussian since the integrand is quadratic in fields. To obtain the generating function for the interacting fields, we first calculate the generating function for the free fields

$$\begin{aligned} Z_0[\mathbf{J}_{EM}, \{\mathbf{J}_\omega\}] &= \int \mathcal{D}[\mathbf{A}]\mathcal{D}[\mathbf{X}] \\ &\quad \times \exp\big[\imath \int d^4x \{\mathcal{L}_{EM} + \mathcal{L}_{mat} + \mathbf{J}_{EM}\cdot\mathbf{A} + \int d\omega\, \mathbf{J}_\omega \cdot \mathbf{X}\}\big]. \end{aligned} \tag{7}$$

Using the 4-dimensional version of Gauss's theorem

$$\int d^4x\, \mathcal{L}_{em} = -\int d^4x\, \big(\mathbf{A}\cdot(\bigtriangledown\times\bigtriangledown\times\mathbf{A}) + \mathbf{A}\cdot\partial_t^2\mathbf{A}\big), \tag{8}$$

and using the integration by parts

$$\int d^4x\ \dot{\mathbf{X}}(\omega,x)\ \cdot\ \dot{\mathbf{X}}(\omega,x) = -\int d^4x\, \mathbf{X}(\omega,x)\ \cdot\ \frac{\partial^2}{\partial t^2}\mathbf{X}(\omega,x). \tag{9}$$

From Eqs. (8) and (9), the free generating functional (7) can be written as

$$\begin{aligned} Z_0[\mathbf{J}_{EM}, \mathbf{J}_\omega] &= \int \mathcal{D}[\mathbf{A}]\mathcal{D}[\mathbf{X}] \exp -\frac{\imath}{2}\big[\int d^4x \{(\mathbf{A}\cdot\bigtriangledown\times\bigtriangledown\times\mathbf{A} + \mathbf{A}\cdot\partial_t^2\mathbf{A}) - \mathbf{J}_{EM}\cdot\mathbf{A} \\ &\quad + \int d\omega \mathbf{X}(\omega,x)(\frac{\partial^2}{\partial t^2} + \omega^2)\mathbf{X}(x,\omega) - \mathbf{J}_\omega(x)\cdot\mathbf{X}(x,\omega)\}\big] \end{aligned} \tag{10}$$

The integral in Eq. (10) can be easily calculated from the field version of the quadratic integral formula and the result is

$$\begin{aligned} Z_0[\mathbf{J}_{EM}, \mathbf{J}(\omega)] &= \exp -\frac{i}{2}\big[\int d^4x \int d^4x' \mathbf{J}_{EM}\cdot \overleftrightarrow{G}^{(0)}_{EM}(x-x')\cdot\mathbf{J}_{EM} \\ &\quad + \int d^3\mathbf{x}\int dt \int dt' \int d\omega \mathbf{J}_\omega(\mathbf{x},t)\cdot\overleftrightarrow{G}_\omega(t-t')\cdot\mathbf{J}_\omega(\mathbf{x},t')\big], \end{aligned} \tag{11}$$

here the space component of the point $x \in \mathbb{R}^4$ is indicated in bold by $\mathbf{x} \in \mathbb{R}^3$ and the time component by t or $x_0 \in \mathbb{R}$. The Green functions $\overleftrightarrow{G}_\omega(t-t')$ and $\overleftrightarrow{G}^{(0)}_{EM}(x-x')$ are the propagators for free fields and satisfy the following equations

$$(\bigtriangledown\times\bigtriangledown\times + \frac{\partial^2}{\partial t^2})\overleftrightarrow{G}^{(0)}_{EM}(x-x') = \delta(x-x') \tag{12}$$

$$\{\frac{\partial^2}{\partial t^2} + \omega^2\}\overleftrightarrow{G}_{\omega\alpha,\beta}(t-t') = \delta(t-t')\delta_{\alpha,\beta}. \tag{13}$$

Taking the Fourier transform of Eqs.(12) and (13) we find

$$\overleftrightarrow{G}^{(0)\parallel}_{EM,\alpha\beta}(\mathbf{k},\omega) = -\frac{k_\alpha k_\beta}{\omega^2} \tag{14}$$

and

$$\overleftrightarrow{G}^{(0)\perp}_{EM,\alpha\beta}(\mathbf{k},\omega) = \frac{\delta_{\alpha\beta} - \widetilde{k}_\alpha \widetilde{k}_\beta}{\mathbf{k}^2 - \omega^2} \tag{15}$$

$$G_{\omega\alpha\beta}(\omega') = \frac{1}{\omega^2 - \omega'^2 - \imath 0^+}\,\delta_{\alpha\beta} \tag{16}$$

where $\widetilde{k}$ is a unit vector along $\mathbf{k}$ and $\overleftrightarrow{G}^{\parallel}$ and $\overleftrightarrow{G}^{\perp}$ refer to the longitudinal and transverse parts of Green's tensor in Fourier space. For further simplicity we define

$$\mathbf{J}_P = \int d\omega \nu^{(1)}(\omega)\mathbf{J}_\omega + \int d\omega \int d\omega' \nu^{(2)}(\omega,\omega')\mathbf{J}_\omega \mathbf{J}_{\omega'} + \int d\omega \int d\omega' \nu^{(3)}(\omega,\omega',\omega'')\mathbf{J}_\omega \mathbf{J}_{\omega'} \mathbf{J}_{\omega''} + \cdots \tag{17}$$

Now the generating functional of the interacting fields can be written in terms of the free generating functional as [34]

$$\begin{aligned}
Z[\mathbf{J}_{EM},\mathbf{J}_P] &= Z^{-1}[0] e^{i\int d^4z \mathcal{L}_{int}\left(\frac{\delta}{\delta \mathbf{J}_{EM}(z)},\frac{\delta}{\delta \mathbf{J}_P(z)}\right)} Z_0[\mathbf{J}_{EM},\mathbf{J}_\omega] \\
&= Z^{-1}[0] \sum_{n=0}^{\infty} \frac{1}{n!}\Bigg\{\imath \int d^4 z \int d\omega_1\, \nu^{(1)}(\omega_1) \frac{\delta}{\delta \mathbf{J}_{EM}(z)} \cdot \frac{\partial}{\partial z_0} \frac{\delta}{\delta \mathbf{J}_{\omega_1}(z)} \\
&+ \int d\omega_1 \int d\omega_2\, \nu^{(2)}(\omega_1,\omega_2) \frac{\delta}{\delta \mathbf{J}_{EM}(z)} \cdot \frac{\partial}{\partial z_0} \frac{\delta}{\delta \mathbf{J}_{\omega_1}(z)} \frac{\delta}{\delta \mathbf{J}_{\omega_2}(z)} \\
&+ \int d\omega_1 \int d\omega_2 \int d\omega_3\, \nu^{(3)}(\omega_1,\omega_2,\omega_3) \frac{\delta}{\delta \mathbf{J}_{EM}(z)} \cdot \frac{\partial}{\partial z_0} \frac{\delta}{\delta \mathbf{J}_{\omega_1}(z)} \frac{\delta}{\delta \mathbf{J}_{\omega_2}(z,)} \frac{\delta}{\delta \mathbf{J}_{\omega_3}(z)} \\
&+ \cdots \Bigg\}^n Z_0[\mathbf{J}_{EM},\mathbf{J}_\omega]
\end{aligned} \tag{18}$$

where $Z[0]$ a normalization factor.

III. LINEAR GREEN TENSORS

In this section we consider a linear medium, so the sequence of nonlinear coupling functions $\nu^{(2)}$, $\nu^{(3)}$, etc. are zero. Following the procedure presented in [29], the linear Green's function of the electromagnetic field can be obtained as

$$\overleftrightarrow{G}^{(1)}_{\alpha\beta}(x-y) = i\frac{\delta^2 Z[J]}{\delta J_\alpha(x)\delta J_\beta(y)}\Big|_{J=0}. \tag{19}$$

To facilitate the calculations we take the time-Fourier-transform of the Green's function (19), after some lengthy but straightforward calculations we find the Green's function in frequency-space as

$$\begin{aligned}\overleftrightarrow{G}^{(1)}_{EM}(\mathbf{x}-\mathbf{x}',\omega) &= \overleftrightarrow{G}_0(\mathbf{x}-\mathbf{x}',\omega) \\ &+ \int d\mathbf{x}_1[\overleftrightarrow{G}_0(\mathbf{x}-\mathbf{x}_1,\omega)\int d\omega'\{\nu^2(\omega')\omega^2\overleftrightarrow{G}_{\omega'}(\omega)\}\overleftrightarrow{G}^{(1)}_{EM}(\mathbf{x}_1-\mathbf{x}',\omega)]. \quad (20)\end{aligned}$$

It can be shown that this Green's function satisfies the following equation

$$\nabla\times\nabla\times\overleftrightarrow{G}^{(1)}_{EM}(\mathbf{x}-\mathbf{x}',\omega)-\omega^2\epsilon(\omega)\overleftrightarrow{G}^{(1)}_{EM}(\mathbf{x}-\mathbf{x}',\omega)=\overleftrightarrow{\delta}(\mathbf{x}-\mathbf{x}') \quad (21)$$

with the following formal solution

$$\overleftrightarrow{G}^{(1)\|}_{EM,\alpha\beta}(\mathbf{k},\omega) = -\frac{k_\alpha k_\beta}{\omega^2\epsilon(\omega)} \quad (22)$$

$$\overleftrightarrow{G}^{(1)\perp}_{EM,\alpha\beta}(\mathbf{k},\omega) = \frac{\delta_{\alpha\beta}-\widetilde{k}_\alpha\widetilde{k}_\beta}{\mathbf{k}^2-\omega^2\epsilon(\omega)} \quad (23)$$

where $\epsilon(\omega)=1+\chi^{(1)}(\omega)$ is the linear permittivity of the medium in frequency domain and the linear susceptibility $\chi^{(1)}(\omega)$ can be written as

$$\chi^{(1)}(\omega) = \int_0^\infty d\omega'\frac{[\nu^{(1)}(\omega')]^2}{\omega'^2-\omega^2-\imath 0^+}. \quad (24)$$

These are complex functions of frequency which satisfy Kramers-Kronig relations and have the properties of response functions i.e, $\epsilon(-w^*)=\epsilon^*(\omega)$ and $Im\epsilon(\omega)>0$ provided that $(\nu^{(1)})^2(-\omega^*)=(\nu^{(1)})^2(\omega)$. It can be shown that these functions have no poles in the upper half plane and tend to unity as $\omega\longrightarrow\infty$. As a consequence, the electric susceptibility of the medium in the time domain can be written as

$$\chi^{(1)}(t)=\begin{cases}\displaystyle\int_0^\infty d\omega\frac{\sin\omega(t)}{\omega}[\nu^{(1)}(\omega)]^2 & t>0\\[2ex] 0 & t<0\end{cases} \quad (25)$$

which is real and causal. This proves that the constitutive relations of the medium, with arbitrary dispersion and absorption, is properly described by the present model. If we are given a definite permittivity for the medium, then we can inverse the relations (24) and find the corresponding coupling function $\nu^{(1)}(\omega)$ as

$$\nu^{(1)}(\omega)=\sqrt{\frac{2\omega}{\pi}Im\chi^{(1)}(\omega)}. \quad (26)$$

In a similar way we can obtain the other correlation functions among different fields [29], but here we consider only the following correlation function

$$\overset{\leftrightarrow}{G}{}^{(1)}_{EM,P}(\mathbf{x}-\mathbf{x}',\omega) = i\omega\chi^{(1)}(\omega)\overset{\leftrightarrow}{G}{}^{(1)}_{EM}(\mathbf{x}-\mathbf{x}',\omega), \tag{27}$$

which shows that $\chi^{(1)}(\omega)$ is a linear electric susceptibility [30, 33]. In this way, we can easily define the nonlinear susceptibilities of the medium via two, three and n-point correlation functions between electromagnetic field and the polarization field [36]. These correlation functions satisfy the fluctuation−dissipation theorem [28, 36, 37]. In the next sections we apply the path-integral formalism to calculate the Green's tensors and Casimir energy in the presence of a nonlinear dielectric medium. To simplify the problem we work in Fourier space and separate the Green's tensor to transverse and longitudinal parts. The longitudinal part of the Green's function does not lead to any force since this part of the Green's function is local and the local fields do not lead to any Casimir force [29], so we only consider the transverse part and for simplicity drop the superscript $^{\perp}$.

IV. NONLINEAR GREEN TENSORS

Now let us consider the contributions induced from a nonlinear medium. To elucidate the method, we start with the term $\nu^{(2)}$ and generalize it to include the higher order terms like $\nu^{(3)}$, $\nu^{(4)}$, etc. In analogue to the linear medium we define the nonlinear Green's function of the electromagnetic field up to the first order of nonlinearity as

$$\overset{\leftrightarrow}{G}{}^{(2)}_{EM}(x-y) = i\frac{\delta^2 Z[J]}{\delta J_{EM}(x)\delta J_{EM}(y)}\Big|_{J=0}. \tag{28}$$

The nonlinear Green's function (28), after some lengthy but straightforward calculations, can be obtained in reciprocal space as follows

$$\begin{aligned}
\overset{\leftrightarrow}{G}{}^{(2)}_{EM}(\mathbf{k},\omega) &= \overset{\leftrightarrow}{G}{}^{(1)}_{EM}(\mathbf{k},\omega) + \overset{\leftrightarrow}{G}{}^{(1)}_{EM}(\mathbf{k},\omega)\Delta^{(2)}_{NL}(\omega)\overset{\leftrightarrow}{G}{}^{(1)}_{EM}(\mathbf{k},\omega) \\
&+ \overset{\leftrightarrow}{G}{}^{(1)}_{EM}(\mathbf{k},\omega)\Delta^{(2)}_{NL}(\omega)\overset{\leftrightarrow}{G}{}^{(1)}_{EM}(\mathbf{k},\omega)\Delta^{(2)}_{NL}(\omega)\overset{\leftrightarrow}{G}{}^{(1)}_{EM}(\mathbf{k},\omega) + ... \\
&= \frac{1}{\mathbf{k}^2-\omega^2[\epsilon(\omega)+\Delta^{(2)}_{NL}(\omega)]},
\end{aligned} \tag{29}$$

where

$$\Delta^{(2)}_{NL}(\omega) = \frac{-\imath}{2\pi}\int_0^\infty d\omega_1\int_0^\infty d\omega_2\int_0^\infty d\omega'[\nu^{(2)}(\omega_1,\omega_2)]^2\frac{1}{\omega_1^2-(\omega-\omega')^2-\imath 0^+}\times\frac{1}{\omega_2^2-\omega'^2-\imath 0^+}, \tag{30}$$

is the first order correction to the Green's function of the electromagnetic field due to the nonlinearity of the medium. In a similar way, we find for $\nu^{(3)}$

$$\overset{\leftrightarrow}{G}{}^{(3)}_{EM}(\mathbf{k},\omega) \; = \; \frac{1}{\mathbf{k}^2 - \omega^2[\epsilon(\omega) + \Delta^{(2)}_{NL}(\omega) + \Delta^{(3)}_{NL}(\omega)]} \tag{31}$$

where

$$\begin{aligned} \Delta^{(3)}_{NL}(\omega) \; &= \; (\frac{-\imath}{2\pi})^2 \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int_0^\infty d\omega_3 \int_0^\infty d\omega'_2 \int_0^\infty d\omega'_3 [\nu^{(3)}(\omega_1,\omega_2,\omega_3)]^2 \\ &\times \frac{1}{\omega_1^2 - (\omega - \omega'_2)^2 - \imath 0^+} \times \frac{1}{\omega_2^2 - \omega'^2_2 - \imath 0^+} \times \frac{1}{\omega_2^2 - \omega'^2_3 - \imath 0^+} \end{aligned} \tag{32}$$

following these procedure we find the nonlinear Green's function of the electromagnetic field up to the n'th order of the nonlinearity as

$$\overset{\leftrightarrow}{G}{}^{(n)}_{EM}(\mathbf{k},\omega) \; = \; \frac{1}{\mathbf{k}^2 - \omega^2[\epsilon(\omega) + \Delta^{(2)}_{NL}(\omega) + \Delta^{(3)}_{NL}(\omega) + \cdots + \Delta^{(n)}_{NL}(\omega)]} \tag{33}$$

where

$$\begin{aligned} \Delta^{(n)}_{NL}(\omega) \; &= \; (\frac{-\imath}{2\pi})^{n-1} \int_0^\infty d\omega_1 \int_0^\infty d\omega'_2 \int_0^\infty d\omega_2 \cdots \int_0^\infty d\omega'_n \int_0^\infty d\omega_n [\nu^{(n)}(\omega_1,\omega_2,\cdots,\omega_n)]^2 \\ &\times \frac{1}{\omega_1^2 - (\omega - \omega'_2 - \cdots - \omega'_n)^2 - \imath 0^+} \times \frac{1}{\omega_2^2 - \omega'^2_2 - \imath 0^+} \times \cdots \times \frac{1}{\omega_n^2 - \omega'^2_n - \imath 0^+}. \end{aligned} \tag{34}$$

In a similar way we can obtain correlation functions among different fields. For example, 3-points Green's function [35] up to the first order of nonlinearity is defined by

$$\begin{aligned} \overset{\leftrightarrow}{G}{}^{(2)}_{EM,EM,P}(\mathbf{x}-\mathbf{x}',\mathbf{x}_1-\mathbf{x}',\omega_1,\omega_2) \; &= \; \frac{\delta^2 Z[J]}{\delta J_{EM}(x)\delta J_{EM}(x_1)\delta J_{\mathbf{P}}(x')}\Big|_{J=0}. \\ &= \; -\omega_1\omega_2\chi^{(2)}(\omega_1,\omega_2)\overset{\leftrightarrow}{G}{}^{(1)}_{EM}(\mathbf{x}-\mathbf{x}',\omega_1)\overset{\leftrightarrow}{G}{}^{(1)}_{EM}(\mathbf{x}_1-\mathbf{x}',\omega_2) \end{aligned} \tag{35}$$

where

$$\chi^{(2)}(\omega_1,\omega_2) = \int_0^\infty d\omega'_1 \int_0^\infty d\omega'_2 \nu^{(2)}(\omega'_1,\omega'_2)\nu^{(1)}(\omega'_1)\nu^{(1)}(\omega'_2)\frac{1}{\omega'^2_1 - \omega_1^2 - \imath 0^+} \times \frac{1}{\omega'^2_2 - \omega_2^2 - \imath 0^+}, \tag{36}$$

is the second order susceptibility. Also we note that in deriving Eq.(35) only leading terms have been kept and very small terms have been ignored. In order to make contact with

standard notation, let us recall the definition of the second order of the polarization within the framework of response theory

$$\mathbf{P}^{(2)}(t) = \int_{-\infty}^{t}\int_{-\infty}^{t} dt\, dt'\, \chi^{(2)}(t-t', t-t'')\mathbf{E}(t')\mathbf{E}(t') \,+ \mathbf{P}_N^{(2)}(t), \tag{37}$$

where $\mathbf{P}_N^{(2)}$, is the noise operator up to the second order of nonlinearity. It is easy to show that Eq.(36) in time-domain can be written as

$$\chi^{(2)}(t_1,t_2) = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \nu^{(2)}(\omega_1,\omega_2)\nu^{(1)}(\omega_1)\nu^{(1)}(\omega_2)\frac{\sin\omega_1 t_1}{\omega_1} \times \frac{\sin\omega_2 t_2}{\omega_2}, \tag{38}$$

which is consistent with the results have been reported in [24]. Following these procedure we find the susceptibility up to the n'th order as follows

$$\begin{aligned}\chi^{(n)}(\omega_1,\cdots,\omega_n) \;&=\; \int_0^\infty d\omega_1'\cdots\int_0^\infty d\omega_n'\nu^{(n)}(\omega_1',\cdots,\omega_n')\nu^{(1)}(\omega_1')\cdots\nu^{(1)}(\omega_n')\\ &\times\frac{1}{\omega_1'^2-\omega_1^2-\imath 0^+}\times\cdots\times\frac{1}{\omega_n'^2-\omega_n^2-\imath 0^+},\end{aligned} \tag{39}$$

which in time-domain can be expressed as

$$\begin{aligned}\chi^{(n)}(t_1,\cdots,t_n) \;&=\; \int_0^\infty d\omega_1\cdots\int_0^\infty d\omega_n\nu^{(n)}(\omega_1,\cdots,\omega_n)\nu^{(1)}(\omega_1)\cdots\nu^{(1)}(\omega_n)\\ &\times\frac{\sin\omega_1 t_1}{\omega_1}\times\cdots\times\frac{\sin\omega_n t_n}{\omega_n}.\end{aligned} \tag{40}$$

Now if we are given definite nth order susceptibility of the medium then we can inverse the relations (40) and using Eq.(26) the corresponding coupling function $\nu^{(n)}$ can be found as

$$\nu^{(n)}(\omega_1,\omega_2,\cdots,\omega_n) = \frac{\sqrt{\omega_1\omega_2\cdots\omega_n}\; Im[\chi^{(n)}(\omega_1,\omega_2,\cdots,\omega_n)]}{(2\pi)^{n/2}\sqrt{Im[\chi^{(1)}(\omega_1)]Im[\chi^{(1)}(\omega_2)]\cdots Im[\chi^{(1)}(\omega_n)]}}, \tag{41}$$

where

$$\begin{aligned}Im\chi^{(n)}(\omega_1,\cdots,\omega_n) \;&=\; \int_0^\infty dt_1\cdots\int_0^\infty dt_n\,\chi^{(n)}(t_1,\cdots,t_n)\sin\omega_1 t_1\cdots\sin\omega_n t_n\\ &= \frac{\omega_1\omega_2\cdots\omega_n}{(2\pi)^n}\int_0^\infty d\omega_1'\cdots\int_0^\infty d\omega_n'\chi^{(n)}(\omega'_1,\cdots,\omega'_n)\\ &\times\frac{1}{\omega_1^2-\omega_1'^2+\imath 0^+}\times\cdots\times\frac{1}{\omega_n^2-\omega_n'^2+\imath 0^+}.\end{aligned} \tag{42}$$

From Eq.(41) it is clear that the nth order coupling function is an odd function in frequencies. Therefore, the nth order coupling function will be an even(odd) function if n is even(odd). Using this symmetry, Eq.(34) can be rewritten as

$$\Delta_{NL}^{(n)}(\omega) \;=\; \frac{2\pi\imath}{(8\pi)^n}\int_0^\infty d\omega_2\cdots\int_0^\infty d\omega_n\frac{\big(Im[\chi^{(n)}(\omega-\omega_2-\cdots-\omega_n,\omega_2,\cdots,\omega_n)]\big)^2}{Im[\chi^{(1)}(\omega-\omega_2-\cdots-\omega_n)]Im[\chi^{(1)}(\omega_2)]\cdots Im[\chi^{(1)}(\omega_n)]}. \tag{43}$$

V. CALCULATING THE CASIMIR FORCE

A. General formalism

In this section we calculate the Casimir force for the simplest configuration consisting of two prefect conducting plates separated by a nonlinear medium of width h. For this purpose we consider the electromagnetic field as transverse magnetic(TM) and transverse electric(TE) modes. These modes are represented by scaler fields which satisfy the Dirichlet

$$\varphi(X_\alpha) = 0 \tag{44}$$

or Neumann

$$\partial_n \varphi(X_\alpha) = 0 \tag{45}$$

boundary conditions where $X_\alpha, (\alpha = 1, 2)$ is an arbitrary point on the conducting plates. To obtain the partition function from the Lagrangian we apply the Wick's rotation, $(t \to \imath\tau)$ and change the signature of space-time from Minkowski to Euclidean. The Dirichlet or Neumann boundary conditions can be represented by auxiliary fields $\psi_\alpha(X_\alpha)$ as [38]

$$\delta(\varphi(X_\alpha)) = \int \mathcal{D}[\psi_\alpha(X_\alpha)] e^{\imath \int \psi(X_\alpha)\varphi(X_\alpha)}, \tag{46}$$

and

$$\delta(\partial_n\varphi(X_\alpha)) = \int \mathcal{D}[\psi_\alpha(X_\alpha)] e^{\imath \int \partial_n\psi(X_\alpha)\varphi(X_\alpha)}. \tag{47}$$

Using Eqs.(46,47) the partition function can be written as

$$Z_D = Z_0^{-1} \int \mathcal{D}[\varphi] \prod_{\alpha=1}^{2} \mathcal{D}[\psi_\alpha(X_\alpha)]) e^{S_D[\varphi]} \tag{48}$$

$$Z_N = Z_0^{-1} \int \mathcal{D}[\varphi] \prod_{\alpha=1}^{2} \mathcal{D}[\psi_\alpha(X_\alpha)]) e^{S_N[\varphi]} \tag{49}$$

where $S_D(\varphi)$ and $S_N(\varphi)$ are defined respectively by

$$S_D[\varphi] = \int d^{(n+1)}x \{\mathcal{L}(\varphi(x)) + \varphi(x) \sum_{\alpha=1}^{2} \int d^{(n)}X \delta(X - X_\alpha)\psi_\alpha(x)\}. \tag{50}$$

and

$$S_N[\varphi] = \int d^{(n+1)}x \{\mathcal{L}(\varphi(x)) + \varphi(x) \sum_{\alpha=1}^{2} \int d^{(n)}X \delta(X - X_\alpha)\partial_n\psi_\alpha(x)\}. \tag{51}$$

By comparing the Eqs.(50, 51) and (6), we can rewrite Eqs. (50, 51) as

$$Z_D = \int \prod_{\alpha=1}^{2} \mathcal{D}[\psi_\alpha(x)] Z(\sum_{\alpha=1}^{2} \int d^n X \delta(X - X_\alpha)\psi_\alpha(X)) \tag{52}$$

and

$$Z_N = \int \prod_{\alpha=1}^{2} \mathcal{D}[\psi_\alpha(x)] Z(\sum_{\alpha=1}^{2} \int d^n X \delta(X - X_\alpha)\partial_n\psi_\alpha(X)) \tag{53}$$

where $Z(\sum_{\alpha=1}^{2} \int d^n X_\alpha \delta(X - X_\alpha)\psi_\alpha(X))$ and $Z(\sum_{\alpha=1}^{2} \int d^n X_\alpha \delta(X - X_\alpha)\partial_n\psi_\alpha(X))$ are the generating functionals of interacting fields defined in Eq.(18) with imaginary time. From Eqs.(52, 53) and (19) the respective partition functions can be written as

$$Z_D = \int \prod_{\alpha=1}^{2} \mathcal{D}[\psi_\alpha(X_\alpha)] e^{\imath S_D(\psi_1,\psi_2)}, \tag{54}$$

and

$$Z_N = \int \prod_{\alpha=1}^{2} \mathcal{D}[\psi_\alpha(X_\alpha)] e^{\imath S_N(\psi_1,\psi_2)}, \tag{55}$$

where

$$\imath S_D(\psi_1, \psi_2) = \imath J_D \, \mathcal{G} \, J_D, \tag{56}$$

and

$$\imath S_N(\psi_1, \psi_2) = \imath J_N \, \mathcal{G} \, J_N, \tag{57}$$

and $J_D(X)$ and $J_N(X)$ are respectively defined by

$$J_D(X) = \int d^n X \, \delta(X - X_\alpha) \, \psi_\alpha(X), \tag{58}$$

$$J_N(X) = \int d^n X \, \delta(X - X_\alpha) \, \partial_n\psi_\alpha(X). \tag{59}$$

We will define the function $\mathcal{G}$ shortly. The partition functions defined by (54) and (55) are calculated straightforwardly [38]

$$Z_D = \frac{1}{\sqrt{\det \Gamma_D(x, y, h)}}, \tag{60}$$

and

$$Z_N = \frac{1}{\sqrt{\det \Gamma_N(x, y, h)}}, \tag{61}$$

where

$$\Gamma_D(x, y, h) = \left[\begin{array}{cc} \mathcal{G}(x - y, 0) & \mathcal{G}(x - y, h) \\ \mathcal{G}(x - y, h) & \mathcal{G}(x - y, 0) \end{array} \right] \tag{62}$$

and

$$\Gamma_N(x,y,h) = \left[\begin{array}{cc} -\partial_z^2\mathcal{G}(x-y,0) & -\partial_z^2\mathcal{G}(x-y,h) \\ -\partial_z^2\mathcal{G}(x-y,h) & -\partial_z^2\mathcal{G}(x-y,0) \end{array}\right] \tag{63}$$

where $\mathcal{G}$ is the Greens function of the fields after a Wick's rotation. Now in order to calculate the Casimir force we define the effective action as

$$S_{eff} = -\imath \ln Z_D[h] \tag{64}$$

from which the Casimir force can be obtained easily as

$$F = \frac{\partial S_{eff}(h)}{\partial h}. \tag{65}$$

It is easy to show that the Dirichelet and Neumann boundary condition are formally the same and when the plates are complete conductors lead to the same result. To save brevity in what follows we only calculate Casimir force for Dirichelet boundary conditions and its generalization to Neumann boundary conditions is straightforward.

B. Linear and nonlinear effects of the medium (T=0)

To calculate the Casimir energy in presence of a nonlinear medium boundary conditions are imposed on the electromagnetic field and the Γ tensor can be written as

$$\Gamma_{EM,EM}(x,y,h) = \left[\begin{array}{cc} \mathcal{G}_{EM,EM}(x-y,0) & \mathcal{G}_{EM,EM}(x-y,h) \\ \mathcal{G}_{EM,EM}(x-y,h) & \mathcal{G}_{EM,EM}(x-y,0) \end{array}\right]. \tag{66}$$

Now to obtain the Casimir force we proceed in Fourier-space since the Γ tensor is diagonal in this space. The Fourier transformation of $\mathcal{G}_{EM,EM}(x-y,h)$ is

$$\begin{aligned} \mathcal{G}_{EM,EM}(p,q,h) &= \int dx \int dy e^{\imath p\cdot x+\imath q\cdot y} G_{EM,EM}(x-y,h) \\ &= \frac{e^{-h\sqrt{[\epsilon(\imath p_0)+\Delta^{(2)}_{NL}(\imath p_0)+\cdots+\Delta^{(n)}_{NL}(\imath p_0)]p_0^2+p_1^2+p_2^2}}}{2\sqrt{[\epsilon(\imath p_0)+\Delta^{(2)}_{NL}(\imath p_0)+\cdots+\Delta^{(n)}_{NL}(\imath p_0)]p_0^2+p_1^2+p_2^2}}(2\pi)^3\delta(p+q), \end{aligned} \tag{67}$$

where $p=(p_0,\mathbf{p})$ and $\mathbf{p}$ is a vector parallel to the conductors, p_0 is the temporal component of p. Thus, in this case, the Casimir force is

$$F_c = -\int \frac{d^3p}{(2\pi)^3}\left[\frac{\mathcal{Q}(p)}{e^{2\mathcal{Q}(p)h}-1}\right], \tag{68}$$

where

$$\mathcal{Q}(p) = \sqrt{\mathbf{p}^2 + p_0^2[\epsilon(\imath p_0) + \Delta_{NL}^{(2)}(\imath p_0) + \cdots + \Delta_{NL}^{(n)}(\imath p_0)]}. \tag{69}$$

It is easy to show that the Casimir force in the presence of a nonlinear medium (71) is similar to the Casimir force in the presence of a linear one, the only difference is in the definition of $\mathcal{Q}(p)$. In the absence of the nonlinearity i.e., $\Delta_{NL}^{(2)}(\omega) = \cdots = \Delta_{NL}^{(n)}(\omega) = 0$, the original Casimir force between two plates in the presence of a linear medium is recovered.

C. Finite temperature

Our considerations so far have been restricted to zero temperature. In fact the temperature corrections to the Casimir force turned out to be negligible in experiments [9], [39]-[40] where the measurements were performed in the separation range $h < 1mm$. But, at $h > 1mm$, as in [8], the temperature corrections make larger contributions to the zero-temperature force between perfect conductors. The generalization of the formalism to this case is straightforward. The inclusion of temperature may be done in the usual manner [3], [41]-[43]. The finite temperature expression, can be found by replacing the frequency integral by a sum over Matsubara frequencies according to the rule

$$\hbar\int_0^\infty \frac{d\xi}{2\pi} f(\imath\xi) \quad \to \quad k_B T \sum_{l=0}^{\infty}{}' f(\imath\xi_l), \qquad\qquad \xi_l = 2\pi k_B T l/\hbar \tag{70}$$

where T and k_B are the temperature and Boltzmann constants and the prime over the summation means the zeroth term should be given half weight as is conventional. Therefore the Casimir force at finite temperature can be expressed as

$$F_c = -\frac{k_B T}{4\pi^2\hbar} \sum_{l=0}^{\infty}{}' \int d^2\mathbf{p}[\frac{\mathcal{Q}(p)}{e^{2\mathcal{Q}(p)h} - 1}] \tag{71}$$

VI. CONCLUSION

Based on a canonical approach and using path-integral techniques, electromagnetic field in a nonlinear dielectric is quantized and Casimir force, in the presence of a nonlinear medium,

at finite temperature, is calculated.

---

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).

[2] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **29**, 94 (1956); E. M. Lifshitz, Sov. Phys.JETP **2**,73 (1956); I. D. Dzyaloshinskii, E. M. Lifshitz and L. P. Pitaevskii, Usp. Fiz. Nauk **73**, 381 (1961); I. D. Dzyaloshinskii, E. M. Lifshitz and L. P. Pitaevskii, Sov. Phys.Usp. **4**, 153 (1961).

[3] J. Schwinger, L. L. DeRaad, Jr. and K. A. Milton, Ann. Phys. (NY) **115**, 1 (1978).

[4] M. S. Toma, Phys. Rev. A **66**, 052103 (2002); Phys. Lett. A **342**, 381 (2005).

[5] C. Raabe, L. Knoll, and D. G. Welsch, Phys. Rev. A **68**, 033810 (2003); C. Raabe and D. G. Welsch, ibid. **71**, 013814 (2005).

[6] M. J. Spaarnay, Physica **24**, 751 (1958).

[7] E. S. Sabisky and C. H. Anderson Phys. Rev. A **7**, 790 (1973).

[8] S. K. Lamoreaux Phys. Rev. Lett. **78**, 5 (1997); Phys. Rev. Lett. **81**, 5475 (1998).

[9] U. Mohideen and A. Roy, Phys. Rev. Lett. **81**, 4549 (1998); A. Roy and U. Mohideen, Phys. Rev. Lett. **82**, 4380 (1999).

[10] K. A. Milton, TheCasimir Effect: Physical Manifestation of Zero-Point Energy (Singapore: World Scientific, 2001).

[11] P. W. Milonni, The Quantum Vacuum (Academic, San Diego, 1994).

[12] M. Krech, The Casimir Effect in Critical Systems (World Scientific, Singapore, 1994).

[13] V. M. Mostepanenko and N. N. Trunov, The Casimir Effect and its Applications (Clarendon Press, Oxford, 1997).

[14] H. B. Chan, V. A. Aksyuk, R. N. Kleiman, D. J. Bishop and F. Capasso, Science. **291**, 1941 (2001); Phys. Rev. Lett. **87**, 211801 (2001).

[15] K. Busch et al., Phys. Rep. **444**, 101 (2007).

[16] J. D. Joannopoulos, R. D. Meade, and J. N. Winn, Photonic Crystals: Molding the Flow of Light (Princeton University Press, Princeton, NJ, 1995).

[17] K. Sakoda, Optical Properties of Photonic Crystals (Springer- Verlag, Berlin-Heidelberg-New York, 2001).

[18] M. Hillery and L. D. Mlodinow, Phys. Rev. A **30**, 1860 (1984).

[19] P. D. Drummond, Phys. Rev. A **42**, 6845 (1990).

[20] I. Abram and E. Cohen, Phys. Rev. A **44**, 500 (1991).

[21] L. M. Duan and G. C. Guo, Phys. Rev. A 56, 925 (1997).

[22] E. Schmidt et al., J. Mod. Opt. **45**, 377 (1998).

[23] S. Scheel and Dirk-Gunnar Welsch, Phys. Rev. Lett. **96**, 073601 (2006).

[24] M. Amooshahi, E. Amooghorban, Ann. Phys. **325**, 1976 (2010).

[25] R. Golestanian and M. Kardar, Phys. Rev. Lett. **78**, 3421 (1997).

[26] R. Golestanian and M. Kardar, Phys. Rev. A **58**, 1713 (1998).

[27] T. Emig, A. Hanke, R. Golestanian, and M. Kardar, Phys. Rev. Lett. **87**, 260402 (2001).

[28] T. Emig, A. Hanke, R. Golestanian, and M. Kardar, Phys. Rev. A **67**, 022114 (2003).

[29] F. Kheirandish, M. Soltani and J. Sarabadani, Phys. Rev. A 81, 052110 (2010).

[30] B. Huttner and S. M. Barnett, Phys. Rev. A **46**, 4306 (1992); Europhys. Lett. **18**, 487 (1992).

[31] L. G. Suttorp, M. Wubs, Phys. Rev. A **70**, 013816 (2004).

[32] F. Kheirandish and M. Amooshahi, Phys. Rev. A **74**, 042102 (2006).

[33] F. Kheirandish and M. Soltani, Phys. Rev. A **78**, 012102 (2008).

[34] L. Ryder, *Quantum Field Theory*, Second ed. (Cabbridge University Press 1996).

[35] S. Jiang and P. Gallion, J. Opt. Soc. Am. B **26**, 902 (2009).

[36] R. Matloob and H. Falinejad, Phys. Rev A **64**, 042102 (2001).

[37] C. Raabe and D.G. Welsch, Phys. Rev. A **71**, 013814 (2005)

[38] J. Sarabadani and M. F. Miri, Phys. Rev. A **74**, 023801 (2006).

[39] A. Roy, C.-Y. Lin, and U. Mohideen, Phys. Rev. D **60**, 111101(R) (1999).

[40] B. W. Harris, F. Chen, and U. Mohideen, Phys. Rev. A **62**, 052109 (2000).

[41] G. D. Mahan, Many Particle Physics, 3rd ed. Plenum punlisher (2000).

[42] E. M. Lifshitz and L. P. Pitaevskii, Statistical Physics, Part 2, Pergamon Press, Oxford, (1980).

[43] J. I. Kapusta, Finite-Temperature Field Theory, Cambridge University Press (1989).